SAMPLE QUESTION PAPER 2
MATHEMATICS
CLASS - XII

Time Allowed: 3 Hours **Max. Marks: 100**

General Instructions:

(i) *All questions are compulsory.*
(ii) *The question paper consists of **26** questions divided into three Sections **A, B** and **C**.*
(iii) *Question No. **1** to **6** in Section **A**are Very Short Answer Type Questions carrying **one mark** each.*
(iv) *Question No. **7** to **19** in Section **B**are Long Answer I Type Questions carrying **four marks** each.*
(v) *Question No. **20** to **26** in Section **C**are Long Answer II Type Questions carrying **six marks** each.*
(vi) *There is no overall choice. However, internal choice has been provided in **4** questions of **four** marks each and **2** questions of **six** marks each. You have to attempt only **one** of the alternatives in all such questions.*
(vii) *Use of calculator is **not** permitted. You may ask for logarithmic tables, if required.*

SECTION – A

1. Find λ if $\left(2\hat{i}+6\hat{j}+14\hat{k}\right)\times\left(\hat{i}-\lambda\hat{j}+7\hat{k}\right)=\vec{0}$

2. Write the integrating factor of $x\frac{dy}{dx}+3y=x^2$

3. For what value of x

$$[x,-1]\begin{bmatrix}1 & 2\\ 2 & 1\end{bmatrix}\begin{bmatrix}x\\ 1\end{bmatrix}=[3]$$

4. Write the area of parallelogram whose diagonals are $\vec{d}_1=2\hat{i}$ and $\vec{d}_2=3\hat{j}$

5. Find the product of the order and degree of the following differential equation.

$$\left(\frac{d^2y}{dx^2}\right)^2-\left(\frac{dy}{dx}\right)^3=y^2$$

6. A= $\begin{bmatrix}3 & 5 & 3\\ 0 & 2 & 7\\ 0 & 0 & 1\end{bmatrix}$

Write the value of A (adj A)

SECTION – B

7. If $y=\sqrt{x+1}-\sqrt{x-1}$

Prove that $\left(x^2-1\right)\frac{d^2y}{dx^2}+x\frac{dy}{dx}-\frac{y}{4}=0$

8. There are 3 families A, B and C. The number of men, women and children in these families are as under.

	Men	Women	Children
Family A	2	3	2
Family A	2	1	4
Family A	3	3	6

Daily Expenses of men, women and children are Rs 200, Rs 300 and Rs 150 respectively. Using matrix multiplication, calculate the daily expenses of each family. What impact does more children in the family create on the society.

9. Solve for x:

$$\sin^{-1}x-\cos^{-1}x=\sin^{-1}(3x-2)$$

OR

If $\cos^{-1}x+\cos^{-1}y\cos^{-1}z=\pi$. Prove that $x^2+y^2+z^2+2xyz=1$

10. Evaluate:

$$\begin{vmatrix}\sqrt{13}+\sqrt{3} & 2\sqrt{5} & \sqrt{5}\\ \sqrt{15}+\sqrt{26} & 5 & \sqrt{10}\\ 3+\sqrt{65} & \sqrt{15} & 5\end{vmatrix}$$

11. Evaluate:

$$\int\frac{\cos 5x+\cos 4x}{1-2\cos 3x}dx$$

OR

$$\int\log\left(x+\sqrt{x^2+a^2}\right)dx$$

12. If $\vec{a}=2\hat{i}+\hat{k},\vec{b}=\hat{i}+\hat{j}+\hat{k}$ and $\vec{c}=4\hat{i}-3\hat{j}+7\hat{k}$ .Find vector $\vec{r}$ which satisfies $\vec{r}\times\vec{b}=\vec{c}\times\vec{b}$ and $\vec{r}\times\vec{a}=0$.

13. Find the distance between two parallel planes:

$$\vec{r}\cdot\left(2\hat{i}-\hat{j}-2\hat{k}\right)=6 \text{ and } \vec{r}\cdot\left(6\hat{i}-3\hat{j}-6\hat{k}\right)=27$$

14. Verify the mean value theorem for the function $f(x)=(x-4)(x-6)(x-8)$ on the interval $[4,10]$

15. Solve: $x\frac{dy}{dx}+y-x+xy\cot x=0$

16. Consider the experiment of tossing a coin. If the coin shows head, toss it again, but if it shows tail, then throw a die. Find the conditional probability of the event 'the die shows a number greater than 4' given that there is at least one tail.

OR

A die is thrown 3 times. If the first throw is fair, then find the probability of getting 15 as a sum.

17. For what value of k is the function continuous at $x=1$

$$f(x)=\begin{cases}(x-1)\quad \tan\frac{\pi x}{2} & x\neq 1\\ k & x=1\end{cases}$$

18. The curve $3y^2=2ax^2+6b$ passes through the point $P(3,-1)$ and the gradient (slope) of the curve at P is −1. Find the value of a and b.

19. Evaluate:

$$\int\left[\tan(\log x)+\sec^2\log x\right]dx$$ **OR**

$$\int\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}dx$$

SECTION – C

20. Find the equation of plane passing through the point (1, 1, 1) and containing the line

$$\vec{r}=\left(-3\hat{i}+\hat{j}+5\hat{k}\right)+\lambda\left(3\hat{i}-\hat{j}-5\hat{k}\right)$$

21. Find the area bounded by the parabola $y^2=x$ and line y + x = 2.

22. Evaluate:

$$\int_0^1\cot^{-1}(1-x+x^2)dx$$

OR

$$\int_{-1}^{2}\left|x^3-x\right|dx$$

23. A bag contains 3 green and 7 white balls. Two balls are drawn one by one at random without replacement. If the second ball drawn is green. What is the probability that the first ball drawn is also green.

24. Let A be the set of all real numbers except −1 i.e. $A=R-\{-1\}$. Let '$\bullet$' is defined on A as $a\bullet b=a+b+ab$ for all a, b $\in$A. Prove that

(i) '$\bullet$' is a binary operation on A
(ii) The given operation is commutative as well as associative.
(iii) Find the identity element.
(iv) Prove that a of A has $\frac{-a}{1+a}$ is inverse.
(v) Find $(2\cdot3)\cdot4$
(vi) Solve the equation $2\cdot(x\cdot5)=4$

25. Find a point on the hypotenuse of a given right angled triangle from which the perpendiculars can be dropped on other sides to form a rectangle of maximum area.

OR

A point on the hypotenuse of a right angled triangle is at distance a and b from the sides show that minimum length of the hypotenuse is $(a^{2/3}+b^{2/3})^{3/2}$

26. A city building code requires that the area of the windows must be at least ⅛ of the area of the walls and roofs of all new building. Construction cost of new building is Rs 3 per square metre of window area and Rs 1 per square metre of wall and roof area. To the nearest square metre what is the largest surface area of new building can have if daily construction cost cannot exceed Rs 1000.

SAMPLE PAPER 2
SOLUTIONS

1. $\lambda = -3$

2. $IF = e^{\int Pdx}$ when $P = \frac{3}{x}$

$IF = x^3$

3. $x = \pm 2$

4. Area of parallelogram $= \frac{1}{2}\left|\vec{d}_1 \times \vec{d}_2\right|$

Area of parallelogram = 3 sq. unit

5. Order = 2; Degree = 2

Product = 4

6. A adj. A = | A | I where| A | = 6

$$A\ (adj.\ A) = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

SECTION – B

7. $y = \sqrt{x+1} - \sqrt{x-1}$ (1)

Differentiating w.r.t. 'x', we get

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x+1}} - \frac{1}{2\sqrt{x-1}}$$

$$\frac{dy}{dx} = \frac{\sqrt{x-1} - \sqrt{x+1}}{2\sqrt{x+1}\sqrt{x-1}}$$

$$2\sqrt{x^2-1}\frac{dy}{dx} = -\left[\sqrt{x-1} - \sqrt{x+1}\right]$$

$$2\sqrt{x^2-1}\frac{dy}{dx} = -y$$

Squaring both sides,

$$4(x^2-1)\left(\frac{dy}{dx}\right)^2 = y^2$$

Again differentiating w.r.t. 'x',

$$4(x^2-1)2\frac{dy}{dx}\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 8x = 2y\frac{dy}{dx}$$

By cancelling $2\frac{dy}{dx} \neq 0$ from both sides, we get,

$$4(x^2-1)\frac{d^2y}{dx^2} + 4x\left(\frac{dy}{dx}\right) = y$$

$$(x^2-1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - \frac{y}{4} = 0$$

8. The matrix P indicating the members of three families A, B and C are

	Men	Women	Children
A	2	3	2
B	2	1	4
C	3	3	6

$$P = \begin{bmatrix} 2 & 3 & 2 \\ 2 & 1 & 4 \\ 3 & 3 & 6 \end{bmatrix}$$

Where matrix T indicating the daily expense of men, women and children

$$T = \begin{bmatrix} 200 \\ 300 \\ 150 \end{bmatrix} \begin{matrix} \text{Men} \\ \text{Women} \\ \text{Children} \end{matrix}$$

Total expense of each family is obtained as

$$PT = \begin{bmatrix} 2 & 3 & 2 \\ 2 & 1 & 4 \\ 3 & 3 & 6 \end{bmatrix}\begin{bmatrix} 200 \\ 300 \\ 150 \end{bmatrix} = \begin{bmatrix} 2\times200+3\times300+2\times150 \\ 2\times300+1\times300+4\times300 \\ 3\times200+3\times300+6\times150 \end{bmatrix}$$

$$= \begin{bmatrix} 400+600+300 \\ 600+300+1200 \\ 600+900+900 \end{bmatrix} = \begin{bmatrix} 1300 \\ 2100 \\ 2400 \end{bmatrix}$$

Expenses of family A = Rs 1300
Expenses of family B = Rs 2100
Expenses of family C = Rs 2400
More children mean more expenditure. So one should plan family in order to keep a check on the expenses.

9. $\sin^{-1} x - \cos^{-1} x = \sin^{-1}(3x-2)$

$$\frac{\pi}{2} - \cos^{-1} x - \cos^{-1} x = \sin^{-1}(3x-2) \qquad \sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$$

$$\frac{\pi}{2} - 2\cos^{-1} x = \sin^{-1}(3x-2)$$

$$\sin\left(\frac{\pi}{2} - 2\cos^{-1} x\right) = 3x-2$$

$$\cos 2\cos^{-1} x = 3x-2$$

Put $x = \cos\theta$

$$\cos 2\theta = 3\cos\theta - 2$$

$$2\cos^2\theta - 1 = 3\cos\theta - 2$$

$$2\cos^2\theta - 3\cos\theta + 1 = 0$$

$$(\cos\theta - 1)(2\cos\theta - 1) = 0$$

$$\cos\theta = 1,\ \cos\ \theta = \frac{1}{2} \Rightarrow x = 1,\ x = \frac{1}{2}$$

Both of these value satisfies the equation.

Hence, x = 1 $\quad x = \frac{1}{2}$

OR

$$\cos^{-1} x + \cos^{-1} y + \cos^{-1} z = \pi$$

$$\cos^{-1} x + \cos^{-1} y = \pi - \cos^{-1} z$$

$$\cos^{-1}\left(xy - \sqrt{1-x^2}\sqrt{1-y^2}\right) = \pi - \cos^{-1} z$$

$$xy - \sqrt{1-x^2}\sqrt{1-y^2} = \cos\left(\pi - \cos^{-1} z\right)$$

$$xy - \sqrt{1-x^2}\sqrt{1-y^2} = -\cos\cos^{-1} z$$

$$xy - \sqrt{1-x^2}\sqrt{1-y^2} = -z$$

$$xy + z = \sqrt{1-x^2}\sqrt{1-y^2}$$

Squaring both side,

$$x^2y^2 + z^2 + 2xyz = (1-x^2)(1-y^2)$$

$$x^2 + y^2 + z^2 + 2xyz = 1$$

10. Taping out $\sqrt{5}$ from C_2, $\sqrt{5}$ from C_3

$$\sqrt{5}\ \sqrt{5} \begin{vmatrix} \sqrt{13}+\sqrt{3} & 2 & 1 \\ \sqrt{26}+\sqrt{15} & \sqrt{5} & \sqrt{2} \\ \sqrt{65}+3 & \sqrt{3} & \sqrt{5} \end{vmatrix}$$

$$\begin{vmatrix} \sqrt{13}+\sqrt{3} & 2 & 1 \\ \sqrt{2}\sqrt{13}+\sqrt{15} & \sqrt{5} & \sqrt{2} \\ \sqrt{5}\sqrt{13}+3 & \sqrt{3} & \sqrt{5} \end{vmatrix}$$

$C_1 \to C_1 - \sqrt{13}\, C_3$

$$\begin{vmatrix} \sqrt{3} & 2 & 1 \\ \sqrt{15} & \sqrt{5} & \sqrt{2} \\ 3 & \sqrt{3} & \sqrt{5} \end{vmatrix}$$

Take $\sqrt{3}$ common from C_1

$$\begin{vmatrix} 1 & 2 & 1 \\ \sqrt{5} & \sqrt{5} & \sqrt{2} \\ \sqrt{3} & \sqrt{3} & \sqrt{5} \end{vmatrix}$$

$C_1 \to C_1 - C_2$

$$\begin{vmatrix} -1 & 2 & 1 \\ 0 & \sqrt{5} & \sqrt{2} \\ 0 & \sqrt{3} & \sqrt{5} \end{vmatrix}$$

By expending along first column

$$-1\begin{vmatrix} \sqrt{5} & \sqrt{2} \\ \sqrt{3} & \sqrt{5} \end{vmatrix} = 1$$

11.

$$\frac{\cos 5x + \cos 4x}{1 - 2\cos 3x} = \frac{2\cos\frac{9x}{2}\cos\frac{x}{2}}{1 - 2\left(2\cos^2\frac{3x}{2} - 1\right)}$$

$$= \frac{2\cos\frac{x}{2}\cos 3\frac{3x}{2}}{3 - 4\cos^2\frac{3x}{2}} = \frac{2\cos\frac{x}{2}\left(4\cos^3\frac{3x}{2} - 3\cos\frac{3x}{2}\right)}{3 - 4\cos^2\left(\frac{3x}{2}\right)}$$

$$= -2\cos\frac{x}{2}\cos\frac{3x}{2} = -\ (\cos 2x + \cos x)$$

$$-\int(\cos 2x + \cos x)\,dx = -\left(\frac{\sin 2x}{2} + \sin x\right) + C$$

OR

$$\int \log(x + \sqrt{a^2 + x^2})dx = \int \log(x + \sqrt{a^2 + x^2})dx \times 1$$

Integrate by part taping $\log(x + \sqrt{a^2 + x^2})$ as first function

$$= \log(x + \sqrt{a^2 + x^2})x - \int x\frac{1}{x + \sqrt{a^2 + x^2}}\left(1 + \frac{2x}{2\sqrt{a^2 + x^2}}\right)dx$$

$$= x\log(x + \sqrt{a^2 + x^2}) - \frac{1}{2}\int\frac{x\ \ dx}{\sqrt{a^2 + x^2}}$$

$$= x\log(x + \sqrt{a^2 + x^2}) - \frac{1}{2}\frac{(x^2 + a^2)^{\frac{1}{2}}}{\frac{1}{2}} + C$$

$$= x\log(x + \sqrt{a^2 + x^2}) - \sqrt{a^2 + x^2} + C$$

12. $\vec{r} \times \vec{b} = \vec{c} \times \vec{b}$

$(\vec{r} - \vec{c}) \times \vec{b} = 0$ but $\vec{b} \neq 0$

$\Rightarrow \vec{r} - \vec{c} \,||\, \vec{b}$ $\vec{r} \neq \vec{c}$ $(\vec{c} \cdot \vec{d} \neq 0)$

$\Rightarrow \vec{r} - \vec{c} = \lambda\vec{b}$ where λ is scalar

$\Rightarrow \vec{r} = \vec{c} + \lambda\vec{b}$

$\vec{r} = 4\hat{i} - 3\hat{j} + 7\hat{k} + \lambda(\hat{i} + \hat{j} + \hat{k})$

$\vec{r} = \hat{i}(4 + \lambda) + \hat{j}(-3 + \lambda) + \hat{k} + (7 + \lambda)$ (i)

$\vec{r} \cdot \vec{a} = 0$

$(4 + \lambda)2 + (7 + \lambda) = 0$

$2\lambda + \lambda + 15 = 0$

$\lambda = -5$

Put value of λ in (i),

$\vec{r} = -\hat{i} - 8\hat{j} + 2\hat{k}$

13. The given planes in cartesian form can be written as

$2x - y - 2z = 6$

$6x - 3y - 6z - 27 = 0$

Distance between two parallel planes

= The distance from point $P(x_1\ y_1\ z_1)$ to the plane $6x - 3y - 6z - 27 = 0$

$$= \left|\frac{6x_1 - 3y_1 - 6z_1 - 27}{\sqrt{6^2 + 3^2 + 6^2}}\right| = \left|\frac{3(2x_1 - y_1 - 2z_1) - 27}{9}\right|$$

$P(x_1\ y_1\ z_1)$ lies on $2x - y - 2z = 6 \Rightarrow 2x_1 - y_1 - 2z_1 = 6$

$$= \left|\frac{3 \times 6 - 27}{9}\right| = |-1| = 1 \text{ unit}$$

14. $f(x) = (x - 4)(x - 6)(x - 8)$

$f(x) = x^3 - 18x^2 + 104x$

Since polynomial is everywhere continuous and differentiable, therefore

f(x) is defined in [4, 10]

f(x) is continuous in [4, 10]

f(x) is differentiable in (4, 10)

$f'(x) = 3x^2 - 36x + 104$

All the condition of mean value theorem are satisfied. So, there must at least one point $c \in (4, 10)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$3c^2 - 36c + 104 = \frac{f(10) - f(4)}{10 - 4} = \frac{48 - 0}{6} = 8$$

$$3c^2 - 36c + 96 = 0$$

$$c^2 - 12c + 32 = 0$$

$$c = \frac{12 \pm \sqrt{16}}{2} = 4, 8$$

Only possible value of $c = 8 \in (4, 10)$

15. $$x\frac{dy}{dx} + y - x + xy\cot x = 0 \qquad x \neq 0$$

Dividing both side by x,

$$\frac{dy}{dx} + \left(\cot x + \frac{1}{x}\right)y = 1$$

This is a linear differential equation in the form of $\frac{dy}{dx} + Py = Q$

$$P = \cot x + \frac{1}{x} \qquad Q = Q$$

$$IF = e^{\int P dx} = e^{\int\left(\cot x + \frac{1}{x}\right)dx} = e^{\log \sin x + \log x} = e^{\log x \sin x}$$

IF = x sin x

$y \times IF = \int Q \times IF\ dx + c$

$$yx\sin x = \int x \sin x\, dx + c$$

$$I = \int x \sin x\, dx$$

$$= x(-\cos x) + \int \cos x\, dx \quad \text{(by using Integration by part)}$$

$$= -x\cos x + \sin x$$

$$yx\sin x = -x\cos x + \sin x + c$$

$$y = -\cot x + \frac{1}{x} + \frac{c}{x \sin x}$$

16. Let S by the sample space of the random experiment, then

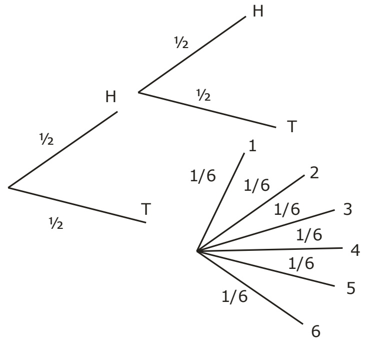
S = [HH, HT, T1, T2, T3, T4, T5, T6]

$P(HH) = \frac{1}{2} \times \frac{1}{2} = 4$ $\qquad$ $P(HT) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

And the probability of each elementary event

T1, T2, T3, T4, T5 and T6 is

$$\frac{1}{2} \times \frac{1}{6} = \frac{1}{12}$$



Note that all the elementary events are not equally likely. Let A be the event of the showing a number greater than 4 and B be the event of that there is at least one tail then

A = [T5, T6] B = [HT, T1, T2, T3, T4, T5, T6]

$A \cap B$ = [T5, T6]

$$P(B) = \frac{1}{4} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12} = \frac{3}{4}$$

$$P(A \cap B) = \frac{1}{12} + \frac{1}{12} = \frac{2}{12} = \frac{1}{6}$$

Required Probability = P(A/B) = $\dfrac{P(A \cap B)}{P(B)} = \dfrac{1/6}{3/4} = \dfrac{2}{9}$

OR

A = getting 15 as the sum in a throw of dice three times

B = getting 4 on first throw

Then B has 1 × 6 × 6 = 36 equally likely cases, out of which two outcomes (4, 5, 6) and (4, 6, 5) are favorable to A.

Required Probability = P(A/B) = Probability of getting 15 as the if there is a 4 on the first throw

$$= \frac{2}{36} = \frac{1}{18}$$

17. Let x = 1 + h, so that when $x \to 1$, $h \to 0$

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} (x-1)\tan \pi\frac{x}{2} = \lim_{h\to 0} h \tan\frac{\pi}{2}(1+h)$$

$$= \lim_{h\to 0} h \tan\left(\frac{\pi}{2} + \frac{\pi}{2}h\right) = \lim_{x\to 0} -h\cot\left(\frac{\pi}{2}h\right)$$

$$= \lim_{h\to 0} -\frac{1}{\dfrac{\tan\frac{\pi}{2}h}{\frac{\pi}{2}h} \times \dfrac{\pi}{2}} = -1 \times \frac{2}{\pi} = -\frac{2}{\pi}$$

Also f(x) = k when x = 1 $\Rightarrow$ f(1) = k

$$\lim_{x\to 1} f(x) = f(1) \Rightarrow \frac{-2}{\pi} = k \Rightarrow \frac{-2}{\pi}$$

18. The given curve is $3y^2 = 2ax^2 + 6b$

Diff. w.r.t. (x),

$$6y\frac{dy}{dx} = 2a \times 2x \Rightarrow \frac{dy}{dx} = \frac{2ax}{3y}$$

Slope of tangent at (3, – 1) is – 1

$$\frac{2a \times 3}{3 \times (-1)} = -1$$

– 2a = – 1

$$a = \frac{1}{2}$$

Also curve passes through the point (3, – 1)

$\Rightarrow$ $3 \times (-1)^2 = 2a \times 3^2 + 6b$

$\Rightarrow$ 3 = 18a + 6b

We know that a = $\frac{1}{2}$ put the value of a in above equation.

3 = 9 + 6b

b = – 1

a = $\frac{1}{2}$ b = – 1 **Ans.**

$$\int\left[\tan\log x+\sec^2(\log x)\right]dx$$

19.

$$\int \tan\log x dx+\int \sec^2(\log x)dx$$

Integrate the first integral by part taking tan (log x) as first integral

$$=\tan(\log x)\cdot x-\int \sec^2 x\cdot dx+\int \sec^2(\log x)dx$$

$$=x\tan(\log x)+c$$

OR

Put x^2 = y only for partial fraction

$$\frac{(x^2+1)}{(x^2+3)}\frac{(x^2+2)}{(x^2+4)}=\frac{(y+1)(y+2)}{(y+3)(y+4)}$$

Let $\frac{(y+1)(y+2)}{(y+3)(y+4)}=1+\frac{A}{(y+3)}+\frac{B}{(y+4)}$

$$(y+1)(y+2)=(y+3)(y+4)+A(y+4)+B(y+3)$$

By equating the of y and constant term

3 = 7 + A + B

2 = 12 + 4A + 3B

By solving, we get, B = – 6, A = 2

$$\frac{(x^2+1)}{(x^2+3)}\frac{(x^2+2)}{(x^2+4)}=1+\frac{2}{x^2+3}-\frac{6}{x^2+4}$$

$$\int\frac{(x^2+1)}{(x^2+3)}\frac{(x^2+2)}{(x^2+4)}dx=\int dx+2\int\frac{dx}{x^2+3}-6\int\frac{dx}{x^2+4}$$

$$x+\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)-\frac{6}{2}\tan^{-1}\left(\frac{x}{2}\right)+c$$

$$x+\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)-3\tan^{-1}\left(\frac{x}{2}\right)+c$$

SECTION – C

20. Equation of plane passing through (1, 1, 1)

a(x – 1) + b(y – 1) + c(z – 1) = 0 .... (1)

If the plane containing given line,

$$\vec{r}=(-3\hat{i}+\hat{j}+5\hat{k})+\lambda(3\hat{i}-\hat{j}-5\hat{k})$$

Plane (1) must passes through (–3, 15) and must be parallel to the given line,

a(– 3 – 1) + b(1 – 1) + c(5 – 1) = 0

– 4a + 0b + 4c = 0 .... (2)

Plane (1) is also parallel to given line.

Normal of plane will be $\perp$ to given line,

3a – b – 5c = 0 .... (3)

By solving (2) and (3),

$$\frac{a}{4}=\frac{-b}{8}=\frac{c}{4}=\lambda$$

a = 4λ b = – 8λ c = 4λ

DR < 1 – 2 1>

Put these values in equation (1),

1(x – 1) – 2(y – 1) + 1(z – 1) = 0

x – 1 – 2y + 2 + z – 1 = 0

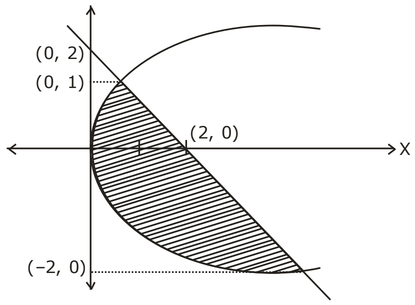
x – 2y + z = 0

which is the required equation of plane.

21. To get point of intersection solve the equation y + x = 2 and y^2 = x

By solving these equations, we get (4, – 2) and (1, 1)

Required shaded area $=\int_{-2}^{1}(2-y)dy-\int_{-2}^{1}y^2dy$



$$= \int_{-2}^{1} (2 - y - y^2)dy = 2y - \frac{y^2}{2} - \frac{y^3}{3}\Big|_{-2}^{1}$$

$$= \left(2 - \frac{1}{2} - \frac{1}{3}\right) - \left(-4 - \frac{4}{2} + \frac{8}{3}\right) = \frac{9}{2} \text{ square unit.}$$

22. $\int_0^1 \cot^{-1}(1\text{-}x\text{+}x^2)dx$

$$= \int_0^1 \tan^{-1}\left(\frac{1}{1 - x + x^2}\right)dx \qquad \left[\cot^{-1} x = \tan^{-1}\frac{1}{x}\right]$$

$$= \int_0^1 \tan^{-1}\left(\frac{x + 1 - x}{1 - x + x^2}\right)dx \qquad \left[\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x + y}{1 - xy}\right)\right]$$

$$= \int_0^1 \tan^{-1} x dx + \int_0^1 \tan^{-1}(1 - x)dx$$

$$= \int_0^1 \tan^{-1} x dx + \int_0^1 \tan^{-1}[1 - (1 - x)]dx \qquad \left[\int_0^9 f(x)dx = \int_0^a f(a - x)dx\right]$$

$$= 2\int_0^1 \tan^{-1} x dx$$

By using integrating by part taking 1 as second function

$$= 2x \tan^{-1} x\Big|_0^1 - \int_0^1 \frac{2x}{1 + x^2}dx = 2 \times \frac{\pi}{4} - \log\left|1 + x^2\right|\Big|_0^1 = \frac{\pi}{2} - \log 2$$

OR

$$\int_{-1}^{2} \left|x^3 - x\right|dx$$

If $x^3 - x = 0 \Rightarrow x(x^2 - 1) = 0 \Rightarrow x(x - 1)(x + 1) = 0$

$x = 0, 1, -1$

Hence [−1 2] is divided into three interval

$x^3 - x \geq 0$ on [− 1, 0]

$x^3 - x \leq 0$ on [0, 1]

$x^3 - x \geq 0$ on [1, 2]

$$\int_{-1}^{2} \left|x^3 - x\right| dx = \int_{-1}^{0} (x^3 - x)dx - \int_{0}^{1} (x^3 - x)dx + \int_{1}^{2} (x^3 - x)dx$$

$$= \left(\frac{x^4}{4} - \frac{x^2}{2}\right)\Bigg|_{-1}^{0} - \left(\frac{x^4}{4} - \frac{x^2}{2}\right)\Bigg|_{0}^{1} + \left(\frac{x^4}{4} - \frac{x^2}{2}\right)\Bigg|_{1}^{2}$$

$$= \frac{11}{4}$$

23. Let E_1, E_2, E_3 be the events as defined below

E_1 = First ball drawn is green

E_2 = First ball drawn is white

A = Second ball drawn is green

$P(E_1) = \frac{3}{10} \qquad P(E_2) = \frac{7}{10}$

$P(A/E_1)$ = Probability of drawing second green ball when one green ball has already been drawn $= \frac{2}{9}$

$P(A/E_2)$ = Probability of drawing second green ball when one white ball has already been drawn $= \frac{3}{9}$

By Bayes Theorem,

$$P\left(\frac{E_1}{A}\right) = \frac{P(E_1)P\left(\frac{A}{E_1}\right)}{P(E_1)P\left(\frac{A}{E_1}\right) + P(E_2)P\left(\frac{A}{E_2}\right)} = \frac{\frac{3}{10}\times\frac{2}{9}}{\frac{3}{10}\times\frac{2}{9}+\frac{7}{10}\times\frac{3}{9}} = \frac{2}{9}$$

24. (i) Consider any $a, b \in A = \quad R - \{-1\}$

$(a \cdot b) = a + b + ab \in R$ but we must show that $a + b + ab \neq -1$

If possible let $a + b + ab = -1$

$\Rightarrow \quad a + b + ab + 1 = 0$

$\Rightarrow \quad (a + 1)(b + 1) = 0$

$\Rightarrow \quad a = -1, b = -1$ which is wrong

$\Rightarrow \quad a$ □ $b \in A$ for all $a, b \in A \Rightarrow$ '□' is a binary operation on A

(ii) For all $a, b, c \in A$, we have

$a * b = a + b + ab = b + a + ba = b * a$

$\Rightarrow$ the given operation is commutative

Also $(a * b) * c = (a + b + ab) * c = (a + b + ab) + c + (a + b + ab)c$

$= a + b + c + ab + bc + ac + abc$

$a * (b * c) = a * (b + c + bc) = a + (b + c + bc) + a(b + c + bc)$

$= a + b + c + ab + bc + ac + abc$

$\Rightarrow \quad a * (b * c) = (a * b) * c \Rightarrow$ the given operation is associative.

(iii) Let $e \in A$ be an identity element of the given operation, then

$e * a = a$, for all $a \in A$

$\Rightarrow \quad e + a + ea = a \Rightarrow e(1 + a) = 0 \Rightarrow e = 0 \qquad (\because a \neq -1)$

We see that $0 * a = 0 + a + 0a = a$; and $a * 0 = a + 0 + a0 = a$

$\Rightarrow$ 0 is an identity element.

(iv) Let b be an inverse of a

$\Rightarrow \quad b * a = e = 0 \Rightarrow b + a + ba = 0$

$\Rightarrow \quad b + ba = -a \Rightarrow b(1 + a) = -a$

$\Rightarrow \quad b = -\frac{a}{1+a} \qquad (\because 1 + a \neq 0)$

$\Rightarrow \quad -\frac{a}{1+a}$ is an inverse of a.

(v) $(2\cdot 3)\cdot 4 = (2 + 3 + 2 \times 3)\cdot 4 = (11\cdot 4)$

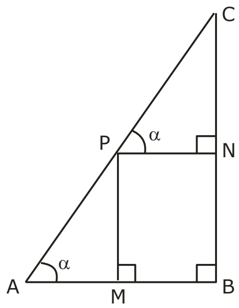
$= (11 + 4 + 11 \times 4) = 15 + 44 = 59$

(vi) $2 * x * 5 = 4 \Rightarrow 2 * (x * 5) = 4 \Rightarrow 2 * (x + 5 + 5x) = 4$

$\Rightarrow \quad 2 + (x + 5 + 5x) + 2(x + 5 + 5x) = 4$

$\Rightarrow \quad 18x + 17 = 4 \Rightarrow x = -\frac{13}{18}$

25. Let □ be the length of the hypotenuse of the given right-angled triangle ABC at B and $\angle CAB = \alpha$ (in radian measure), $0 < \alpha < \frac{\pi}{2}$. As □ABC is given, □ and α are fixed i.e. constant.



Let P be a point on the hypotenuse AC and AP = x, then PC = $- x$.

Let PM and PN be perpendiculars to the other sides, then

MP = x sin α and NP = $(- x)$ cos α

Let A be the area of the rectangle MBNP then

A = MP × NP = x sin α $(- x)$ cos α

$= (x - x^2)$ sin α cos α

Diff. it w.r.t. x, we get

$$\frac{dA}{dx} = (\ell - 2x)\sin\alpha\cos\alpha, \text{ and}$$

$$\frac{d^2A}{dx^2} = (0 - 2.1)\sin\alpha\cos\alpha = -\sin 2\alpha.$$

Now $\frac{dA}{dx} = 0 \Rightarrow (\ell - 2x)\sin\alpha\cos\alpha = 0 \Rightarrow \ell - 2x = 0 \Rightarrow x = \frac{\ell}{2}$.

Also $\left(\frac{d^2A}{dx^2}\right)_{x=\frac{1}{2}} = -2\sin 2\alpha < 0$ $\qquad \left(\because 0 < \alpha < \frac{\pi}{2}, \sin 2\alpha > 0\right)$

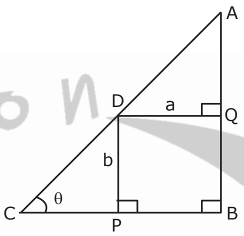
$\Rightarrow$ A is maximum when $x = \frac{\ell}{2}$.

Hence, the area of the rectangle will be maximum when the point is mid-point of the hypotenuse.

OR

Let d be the length of the hypotenuse of rt. $\angle$d = ABC. Let θ (in radian measure) be the angle between the hypotenuse and the base of the triangle, $0 < \theta < \frac{\pi}{2}$. Let D be the point on the hypotenuse AC such that DP = b and DQ = a, then

$\therefore$ CD = b cosecθ and DA = a secθ.



From fig., AC = DA + CD

$\Rightarrow$ d = a secθ + b cosecθ .... (i)

$$\frac{d\ell}{d\theta} = a\sec\theta\tan\theta - b\,\text{cosec}\,\theta\cot\theta \qquad \text{........ (ii)}$$

$$= a\frac{\sin\theta}{\cos^2\theta} - \frac{b\cos\theta}{\sin^2\theta} = \frac{a\sin^3\theta - b\cos^3\theta}{\cos^2\theta\sin^2\theta}$$

Now $\frac{d\ell}{d\theta} = 0 \Rightarrow \frac{a\sin^3\theta - b\cos^3\theta}{\cos^2\theta\sin^2\theta} = 0$

$\Rightarrow$ $a\sin^3\theta = b\cos^3\theta$ $\qquad (\cos^2\theta\sin^2\theta > 0, \text{ as } 0 < \theta < \frac{\pi}{2})$

$\Rightarrow$ $\tan^3\theta = \frac{b}{a} \Rightarrow \tan\theta = \frac{b^{1/3}}{a^{1/3}}$

$$\frac{d^2\ell}{d\theta^2} = \frac{d}{d\theta}(a\sec\theta\tan\theta - b\,\text{cosec}\,\theta\cot\theta) \qquad \text{(using(ii))}$$

$$= a(\sec^3\theta + \sec\theta\tan^2\theta) + b(\text{cosec}^3\theta + \text{cosec}\,\theta\cot^2\theta)$$

$$= a\sec\theta(\sec^2\theta + \tan^2\theta) + b\,\text{cosec}\,\theta(\text{cosec}^2\theta + \cot^2\theta) > 0 \qquad \left(\because 0 < \theta < \frac{\pi}{2}\right)$$

$\therefore$ Length of hypotenuse will be minimum when $\tan\theta = \frac{b^{1/3}}{a^{1/3}}$

Putting $\tan\theta = \frac{b^{1/3}}{a^{1/3}}$ in (i), we get d = $a(1 + \tan^2\theta)^{1/2} + b(1 + \cot^2\theta)^{1/2}$

$$= a\left(1 + \frac{b^{2/3}}{a^{2/3}}\right)^{1/2} + b\left(1 + \frac{a^{2/3}}{b^{2/3}}\right)^{1/2} = a\left(\frac{a^{2/3} + b^{2/3}}{a^{2/3}}\right)^{1/2} + b\left(\frac{b^{2/3} + a^{2/3}}{b^{2/3}}\right)^{1/2}$$

$$= a^{2/3}(a^{2/3} + b^{2/3})^{1/2} + b^{2/3}(b^{2/3} + a^{2/3})^{1/2}$$

$$= (a^{2/3} + b^{2/3})(a^{2/3} + b^{2/3})^{1/2} = (a^{2/3} + b^{2/3})^{3/2}$$

Hence, the minimum length of the hypotenuse = $(a^{2/3} + b^{2/3})^{3/2}$.

26. Let x be number of square metres of area of windows and y be number of square metres of walls and roofs. Then total surface area = x + y

Max. z = x + y subject to constraint

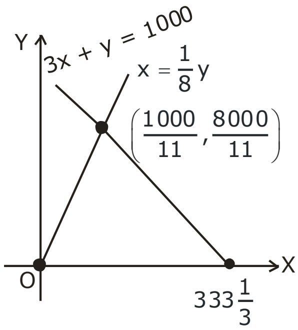
$$x \geq \frac{y}{8}$$

$$3x + y \leq 1000$$

$$x \geq 0, \; y \geq 0$$

LINEAR PROGRAMMING

The graph of the given linear inequalities is shown shaded in the figure given below. Note that feasible region is bounded.



At the three corners of the feasible region:

$$f(0,0) = 0, f\left(333\frac{1}{3}, 0\right) = 333\frac{1}{3}, f\left(\frac{1000}{11}, \frac{8000}{11}\right) = \frac{9000}{11} \approx 818 \text{ sq.m.}$$